IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

Mathematics
Higher level

Specimen paper 1, paper 2 and paper 3

For first examinations in 2006

CONTENTS

IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

MATHEMATICS
HIGHER LEVEL
PAPER 1

SPECIMEN

2 hours

Candidate session number

0	0							

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- Answer all the questions in the spaces provided.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. All students should therefore be advised to show their working. Working may be continued below the lines, if necessary.

1. The polynomial $f(x) = x^3 + 3x^2 + ax + b$ leaves the same remainder when divided by $(x-2)$ as when divided by $(x+1)$. Find the value of a.

2. The displacement s metres of a moving body B from a fixed point O at time t seconds is given by

$$s = 50t - 10t^2 + 1000 .$$

(a) Find the velocity of B in m s^{-1}.

(b) Find its maximum displacement from O.

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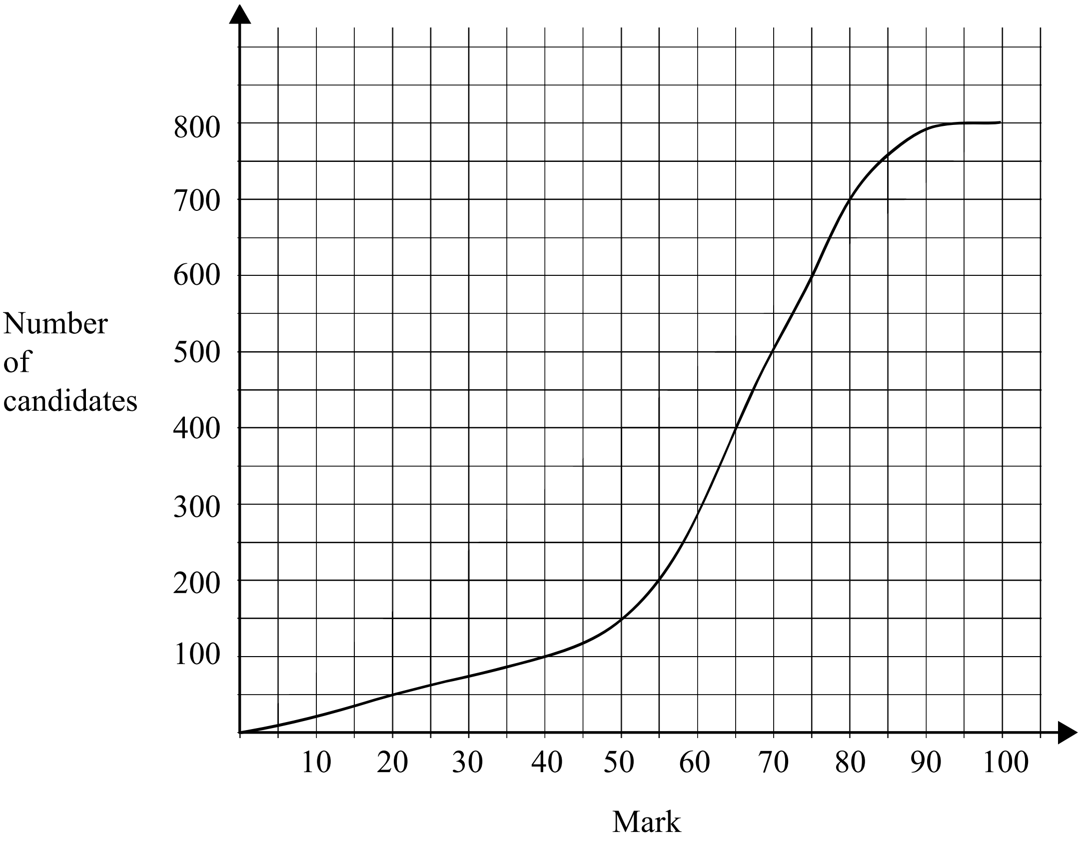
3. A test marked out of 100 is written by 800 students. The cumulative frequency graph for the marks is given below.



(a) Write down the number of students who scored 40 marks or less on the test.

(b) The middle 50 % of test results lie between marks a and b, where $a < b$. Find a and b.

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4. The angle θ satisfies the equation $2\tan^2\theta - 5\sec\theta - 10 = 0$, where θ is in the second quadrant. Find the **exact** value of $\sec\theta$.

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5. A discrete random variable X has its probability distribution given by

$$\mathrm{P}(X = x) = k(x+1), \text{where } x \text{ is } 0, 1, 2, 3, 4.$$

(a) Show that $k = \frac{1}{15}$.

(b) Find $\mathrm{E}(X)$.

6. The function f' is given by $f'(x) = 2\sin\left(5x - \frac{\pi}{2}\right)$.

(a) Write down $f''(x)$.

(b) Given that $f\left(\frac{\pi}{2}\right) = 1$, find $f(x)$.

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7. A sum of $\$\,5\,000$ is invested at a compound interest rate of 6.3 % per annum.

(a) Write down an expression for the value of the investment after *n* full years.

(b) What will be the value of the investment at the end of five years?

(c) The value of the investment will exceed $\$\,10\,000$ after *n* full years.

(i) Write an inequality to represent this information.

(ii) Calculate the minimum value of *n*.

8. The speeds of cars at a certain point on a straight road are normally distributed with mean μ and standard deviation σ. 15 % of the cars travelled at speeds greater than 90 $\text{km}\,\text{h}^{-1}$ and 12 % of them at speeds less than 40 $\text{km}\,\text{h}^{-1}$. Find μ and σ.

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9. The functions *f* and *g* are defined by $f : x \mapsto \mathrm{e}^x$, $g : x \mapsto x + 2$.

(a) Calculate $f^{-1}(3) \times g^{-1}(3)$.

(b) Show that $(f \circ g)^{-1}(3) = \ln 3 - 2$.

10. Given that $z = (b + \mathrm{i})^2$, where b is real and positive, find the **exact** value of b when $\arg z = 60^\circ$.

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11. Find the gradient of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point $(1, -2)$.

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12. A triangle has its vertices at $A(-1, 3, 2)$, $B(3, 6, 1)$ and $C(-4, 4, 3)$.

(a) Show that $\overrightarrow{AB} \bullet \overrightarrow{AC} = -10$.

(b) Show that, to three significant figures, $\cos B\hat{A}C = -0.591$.

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13. **(**(a) Write down the inverse of the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & -3 & 1 \\ 2 & 2 & -1 \\ 1 & -5 & 3 \end{pmatrix}$$

(b) **Hence,** find the point of intersection of the three planes.

$$\begin{aligned} x - 3y + z &= 1 \\ 2x + 2y - z &= 2 \\ x - 5y + 3z &= 3 \end{aligned}$$

(c) A fourth plane with equation $x + y + z = d$ passes through the point of intersection. Find the value of d .

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14. Use the substitution $u = x + 2$ to find $\int \frac{x^3}{(x+2)^2}\,\mathrm{d}x$.

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15. There are 30 students in a class, of which 18 are girls and 12 are boys. Four students are selected at random to form a committee. Calculate the probability that the committee contains

(a) two girls and two boys;

(b) students all of the same gender.

16. The line L is given by the parametric equations $x = 1 - \lambda$, $y = 2 - 3\lambda$, $z = 2$. Find the coordinates of the point on L which is nearest to the origin.

17. The random variable X has a Poisson distribution with mean 4. Calculate

(a) $\mathrm{P}(3 \le X \le 5)$;

(b) $\mathrm{P}(X \ge 3)$;

(c) $\mathrm{P}(3 \le X \le 5 | X \ge 3)$.

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18. Let $f(x) = \dfrac{x+4}{x+1}$, $x \neq -1$ and $g(x) = \dfrac{x-2}{x-4}$, $x \neq 4$. Find the set of values of x such that $f(x) \leq g(x)$.

19. Solve the differential equation $x\dfrac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 1$, given that $y = 0$ when $x = 2$. Give your answer in the form $y = f(x)$.

20. The square matrix $\boldsymbol{X}$ is such that $\boldsymbol{X}^3 = 0$. Show that the inverse of the matrix $(\boldsymbol{I} - \boldsymbol{X})$ is $\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$.

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IB DIPLOMA PROGRAMME
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SPEC/5/MATHL/HP1/ENG/TZ0/XX/M

MARKSCHEME

SPECIMEN PAPER

MATHEMATICS

Higher Level

Paper 1

11 pages

Markscheme Instructions

A. Abbreviations

M Marks awarded for attempting to use a correct **Method:** the working must be seen.

(M) Marks awarded for **Method**: this may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**, usually dependent on preceding ***M*** marks: the working must be seen.

(A) Marks awarded for an **Answer** or for **Accuracy**: this may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**

N Marks awarded for **correct** answers, if **no** working (or no relevant working) shown: in general, these will not be all the marks for the question. Examiners should only award these ***N*** marks for correct answers where there is no working (or if there is working which earns no other marks).

B. Using the markscheme

Follow through (ft) marks: Only award **ft** marks when a candidate uses an incorrect answer in a subsequent **part**. Any exceptions to this will be noted on the markscheme. Follow through marks are now the exception rather than the rule within a question or part question. Follow through marks may only be awarded to work that is seen. Do **not** award ***N*** **(ft)** marks. If the question becomes much simpler then use discretion to award fewer marks.
If a candidate mis-reads data from the **question** apply follow-through.

Discretionary (d) marks: There will be rare occasions where the markscheme does not cover the work seen. In such cases, **(d)** should used to indicate where an examiner has used discretion. It must be accompanied by a brief note to explain the decision made.

It is important to understand the difference between "**implied**" marks, as indicated by the brackets, and marks which can only be awarded for work seen - no brackets. The implied marks can only be awarded if **correct** work is seen or implied in subsequent working. Normally this would be in the next line.

Where ***M1 A1*** are awarded on the same line, this usually means ***M1*** for an attempt to use an appropriate formula, ***A1*** for correct substitution.

As ***A*** marks are normally **dependent** on the preceding ***M*** mark being awarded, it is not possible to award ***M0 A1.***

As ***N*** marks are only awarded when there is no working, it is not possible to award a mixture of ***N*** and other marks.

Accept all correct alternative methods, even if not specified in the markscheme Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative (part) solutions, are indicated by **EITHER**….**OR**. Where possible, alignment will also be used to assist examiners to identify where these alternatives start and finish.

Unless the question specifies otherwise, accept **equivalent forms**. On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer The markscheme indicate the required answer, by allocating full marks at that point. Once the correct answer is seen, ignore further working, unless it contradicts the answer.

Brackets will also be used for what could be described as the well-expressed answer, but which candidates may not write in examinations. Examiners need to be aware that the marks for answers should be awarded for the form preceding the brackets eg in differentiating $f(x) = 2\sin(5x-3)$, the markscheme says

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textit{A1}$$

This means that the ***A1*** is awarded for seeing $\left(2\cos(5x-3)\right)5$, although we would normally write the answer as $10\cos(5x-3)$.

As this is an international examination, all **alternative forms of notation** should be accepted.

Where the markscheme specifies ***M2***, ***A3***, etc, for an answer do NOT split the marks unless otherwise instructed.

Do **not** award full marks for a correct answer, all working must be checked.

Candidates should be penalized **once IN THE PAPER** for an accuracy error (**AP).** There are two types of accuracy error:

- **Rounding errors:** only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule is *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figure*s.

1.

Attempting to find $f(2) = 8 + 12 + 2a + b$	***(M1)***	
$= 2a + b + 20$	***A1***	
Attempting to find $f(-1) = -1 + 3 - a + b$.	***(M1)***	
$= 2 - a + b$	***A1***	
Equating $2a + 20 = 2 - a$	***A1***	
$a = -6$.	***A1***	***N2***

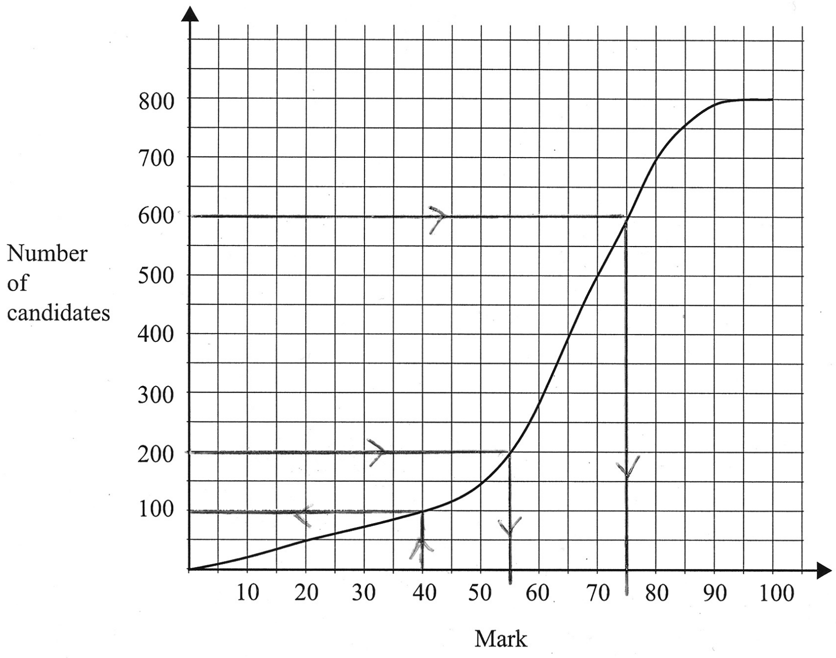
2. (a)

$s = 50t - 10t^2 + 1000$		
$v = \frac{\mathrm{d}s}{\mathrm{d}t}$	***(M1)***	
$= 50 - 20t$	***A1***	***N2***

(b)

Displacement is max when $v = 0$,	***M1***	
ie when $t = \frac{5}{2}$.	***A1***	
Substituting $t = \frac{5}{2}$, $s = 50 \times \frac{5}{2} - 10 \times \left(\frac{5}{2}\right)^2 + 1000$	***(M1)***	
$s = 1062.5$ m	***A1***	***N2***

3. (a)



Lines on graph	***(M1)***	
100 students score 40 marks or fewer.	***A1***	***N2***

(b)

Identifying 200 **and** 600	***A1***	
Lines on graph.	***(M1)***	
$a = 55$, $b = 75$.	***A1 A1***	***N1N1***

4. $2\tan^2\theta - 5\sec\theta - 10 = 0$

Using $1+\tan^2\theta = \sec^2\theta$, $\Rightarrow 2\left(\sec^2\theta - 1\right) - 5\sec\theta - 10 = 0$ ***(M1)***

$2\sec^2\theta - 5\sec\theta - 12 = 0$ ***A1***

Solving the equation eg $\left(2\sec\theta + 3\right)\left(\sec\theta - 4\right) = 0$ ***(M1)***

$\sec\theta = -\frac{3}{2}$ or $\sec\theta = 4$ ***A1***

θ in second quadrant $\Rightarrow \sec\theta$ is negative ***(R1)***

$\Rightarrow \sec\theta = -\frac{3}{2}$ ***A1*** ***N3***

5. (a) Using $\sum \mathrm{P}(X = x) = 1$ ***(M1)***

$\therefore k\times 1 + k\times 2 + k\times 3 + k\times 4 + k\times 5 = 15k = 1$ ***M1A1***

$k = \frac{1}{15}$ ***AG*** ***N0***

(b) Using $\mathrm{E}(X) = \sum x\,\mathrm{P}(X = x)$ ***(M1)***

$= 0\times\frac{1}{15} + 1\times\frac{2}{15} + 2\times\frac{3}{15} + 3\times\frac{4}{15} + 4\times\frac{5}{15}$ ***A1***

$= \frac{8}{3}\left(2\frac{2}{3},\ 2.67\right)$ ***A1*** ***N2***

6. (a) Using the chain rule $f''(x) = \left(2\cos\left(5x - \frac{\pi}{2}\right)\right)5$ ***(M1)***

$= 10\cos\left(5x - \frac{\pi}{2}\right)$ ***A1*** ***N2***

(b) $f(x) = \int f'(x)\,\mathrm{d}x$

$= -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + c$ ***A1***

Substituting to find *c*, $f\left(\frac{\pi}{2}\right) = -\frac{2}{5}\cos\left(5\left(\frac{\pi}{2}\right) - \frac{\pi}{2}\right) + c = 1$ ***M1***

$c = 1 + \frac{2}{5}\cos 2\pi = 1 + \frac{2}{5} = \frac{7}{5}$ ***(A1)***

$f(x) = -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + \frac{7}{5}$ ***A1*** ***N2***

7. (a) $5000(1.063)^n$ ***A1*** ***N1***

(b) Value $= \$5000(1.063)^5 \quad (= \$6786.3511\ldots)$

$= \$6790$ to 3 sf (Accept \$6786, or \$6786.35) ***A1*** ***N1***

(c) (i) $5000(1.063)^n > 10000 \quad (\text{or } (1.063)^n > 2)$ ***A1*** ***N1***

(ii) Attempting to solve the above inequality $n\log(1.063) > \log 2$ ***(M1)***

$n > 11.345...$ ***(A1)***

12 years ***A1*** ***N3***

> **Note:** Candidates are likely to use TABLE or LIST on a GDC to find *n.* A good way of communicating this is suggested below.

Let $y = 1.063^x$ ***(M1)***

When $x = 11, y = 1.9582$, when $x = 12, y = 2.0816$ ***(A1)***

$x = 12$ ie 12 years ***A1*** ***N3***

8. $\mathrm{P}(X > 90) = 0.15$ **and** $\mathrm{P}(X < 40) = 0.12$ ***(M1)***

Finding standardized values 1.036, -1.175 ***A1 A1***

Setting up the equations $1.036 = \dfrac{90-\mu}{\sigma}$, $-1.175 = \dfrac{40-\mu}{\sigma}$ ***(M1)***

$\mu = 66.6$, $\sigma = 22.6$ ***A1 A1*** ***N2N2***

9. (a) $f : x \mapsto \mathrm{e}^x \Rightarrow f^{-1} : x \mapsto \ln x$

$\Rightarrow f^{-1}(3) = \ln 3$ ***A1***

$g : x \mapsto x + 2 \Rightarrow g^{-1} : x \mapsto x - 2$

$\Rightarrow g^{-1}(3) = 1$ ***A1***

$f^{-1}(3) \times g^{-1}(3) = \ln 3$ ***A1*** ***N1***

(b) $f \circ g(x) = f(x+2) = \mathrm{e}^{x+2}$ ***A1***

$\mathrm{e}^{x+2} = 3 \Rightarrow x + 2 = \ln 3$ ***M1A1***

$x = \ln 3 - 2$ ***AG*** ***N0***

10. **METHOD 1**

since $b > 0$ ***(M1)***

$\Rightarrow \arg(b+\mathrm{i}) = 30^\circ$ ***A1***

$\frac{1}{b} = \tan 30^\circ$ ***M1A1***

$b = \sqrt{3}$ ***A2*** ***N2***

METHOD 2

$\arg(b+\mathrm{i})^2 = 60^\circ \Rightarrow \arg(b^2 - 1 + 2b\mathrm{i}) = 60^\circ$ ***M1***

$\frac{2b}{(b^2-1)} = \tan 60^0 = \sqrt{3}$ ***M1A1***

$\sqrt{3}b^2 - 2b - \sqrt{3} = 0$ ***A1***

$(\sqrt{3}b+1)(b-\sqrt{3}) = 0$

since $b > 0$ ***(M1)***

$b = \sqrt{3}$ ***A1*** ***N2***

11. Attempting to differentiate implicitly ***(M1)***

$3x^2y + 2xy^2 = 2 \Rightarrow 6xy + 3x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y^2 + 4xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***A1***

Substituting $x = 1$ and $y = -2$ ***(M1)***

$-12 + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 8 - 8\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***A1***

$\Rightarrow -5\frac{\mathrm{d}y}{\mathrm{d}x} = 4 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{4}{5}$ ***A1***

Gradient of normal is $\frac{5}{4}$. ***A1*** ***N3***

12. (a) Finding **correct** vectors $\overrightarrow{\mathrm{AB}} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ $\overrightarrow{\mathrm{AC}} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$ ***A1 A1***

Substituting correctly in scalar product $\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{AC}} = 4(-3) + 3(1) - 1(1)$ ***A1***

$= -10$ ***AG*** ***N0***

(b) $\left|\overrightarrow{\mathrm{AB}}\right| = \sqrt{26}$ $\left|\overrightarrow{\mathrm{AC}}\right| = \sqrt{11}$ ***(A1)(A1)***

Attempting to use scalar product formula , $\cos \mathrm{B\hat{A}C} = \frac{-10}{\sqrt{26}\sqrt{11}}$ ***M1***

$= -0.591$ (to 3 s.f.) ***AG*** ***N0***

13. (a) $A^{-1} = \begin{pmatrix} 0.1 & 0.4 & 0.1 \\ -0.7 & 0.2 & 0.3 \\ -1.2 & 0.2 & 0.8 \end{pmatrix}$ ***A2*** ***N2***

(b) For attempting to calculate $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ***M1***

$x = 1.2,\ y = 0.6,\ z = 1.6$ (So the point is $(1.2,\ 0.6,\ 1.6)$) ***A2*** ***N2***

(c) $(1.2,\ 0.6,\ 1.6)$ lies on $x + y + z = d$

$\therefore d = 3.4$ ***A1*** ***N1***

14. Substituting $u = x + 2 \Rightarrow u - 2 = x,\ \mathrm{d}u = \mathrm{d}x$ ***(M1)***

$\int \frac{x^3}{(x+2)^2}\mathrm{d}x = \int \frac{(u-2)^3}{u^2}\mathrm{d}u$ ***A1***

$= \int \frac{u^3 - 6u^2 + 12u - 8}{u^2}\mathrm{d}u$ ***A1***

$= \int u\,\mathrm{d}u + \int (-6)\,\mathrm{d}u + \int \frac{12}{u}\mathrm{d}u - \int 8u^{-2}\,\mathrm{d}u$ ***A1***

$= \frac{u^2}{2} - 6u + 12\ln|u| + 8u^{-1} + c$ ***A1***

$= \frac{(x+2)^2}{2} - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + c$ ***A1*** ***N0***

15. (a) Total number of ways of selecting 4 from 30 $= \binom{30}{4}$ ***(M1)***

Number of ways of choosing 2B 2G $= \binom{12}{2}\binom{18}{2}$ ***(M1)***

$\mathrm{P(2B\ or\ 2G)} = \frac{\binom{12}{2}\binom{18}{2}}{\binom{30}{4}} = 0.368$ ***A1*** ***N2***

(b) Number of ways of choosing 4B $= \binom{12}{4}$, choosing 4G $= \binom{18}{4}$ ***A1***

$\mathrm{P(4B\ or\ 4G)} = \frac{\binom{12}{4} + \binom{18}{4}}{\binom{30}{4}}$ ***(M1)***

$= 0.130$ ***A1*** ***N2***

16. **EITHER**

Let s be the distance from the origin to a point on the line, then

(M1)

$$s^2 = (1-\lambda)^2 + (2-3\lambda)^2 + 4$$

$$= 10\lambda^2 - 14\lambda + 9$$ ***A1***

$$\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 20\lambda - 14$$ ***A1***

For minimum $\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 0, \Rightarrow \lambda = \frac{7}{10}$ ***A1***

OR

The position vector for the point nearest to the origin is perpendicular to the direction of the line. At that point:

$$\begin{pmatrix} 1-\lambda \\ 2-3\lambda \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} = 0$$ ***(M1)A1***

Therefore, $10\lambda - 7 = 0$ ***A1***

Therefore, $\lambda = \frac{7}{10}$ ***A1***

THEN

$x = \frac{3}{10}, y = -\frac{1}{10}$ ***(A1)(A1)***

The point is $\left(\frac{3}{10}, \frac{-1}{10}, 2\right)$. ***N3***

17. (a) $\mathrm{P}(3 \le X \le 5) = \mathrm{P}(X \le 5) - \mathrm{P}(X \le 2)$ ***(M1)***

$= 0.547$ ***A1*** ***N2***

(b) $\mathrm{P}(X \ge 3) = 1 - \mathrm{P}(X \le 2)$ ***(M1)***

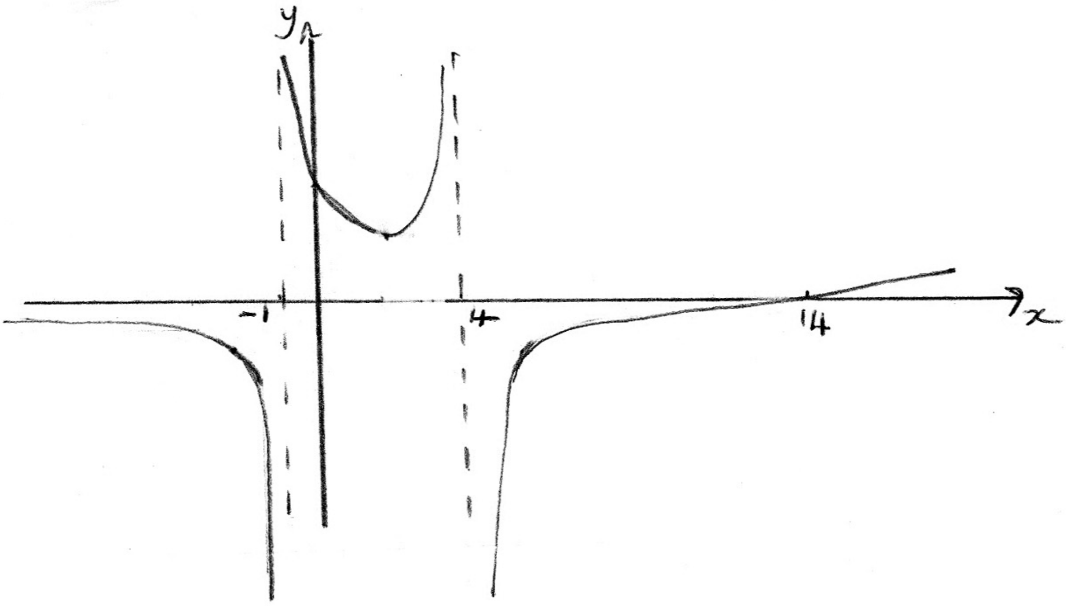
$= 0.762$ ***A1*** ***N2***

(c) $\mathrm{P}(3 \le X \le 5 | X \ge 3) = \frac{\mathrm{P}(3 \le X \le 5)}{P(X \ge 3)} \left(= \frac{0.547}{0.762}\right)$ ***(M1)***

$= 0.718$ ***A1*** ***N2***

18. **METHOD 1**

Graph of $f(x)-g(x)$ ***M1***



A1A1A1

Note: Award ***A1*** for each branch.

$x<-1$ or $4<x\leq 14$ ***A1 A1 N3***

Note: Each value and inequality sign must be correct.

METHOD 2

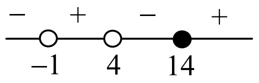
$$\frac{x+4}{x+1}-\frac{x-2}{x-4}\leq 0$$ ***M1***

$$\frac{x^2-16-x^2+x+2}{(x+1)(x-4)}\leq 0$$

$$\frac{x-14}{(x+1)(x-4)}\leq 0$$ ***A1***

Critical value of $x=14$ ***A1***

Other critical values $x=-1$ and $x=4$ ***A1***



$x<-1$ or $4<x\leq 14$ ***A1 A1 N3***

Note: Each value and inequality sign must be correct.

19. $x\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 1, \Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 1$

Separating variables

$\frac{\mathrm{d}y}{y^2+1} = \frac{\mathrm{d}x}{x}$ ***(M1)*** ***A1***

$\arctan y = \ln x + c$ ***A1A1***

$y = 0,\ x = 2 \Rightarrow \arctan 0 = \ln 2 + c$

$-\ln 2 \quad = c$ ***(A1)***

$\arctan y = \ln x - \ln 2 = \ln\frac{x}{2}$

$y = \tan\left(\ln\frac{x}{2}\right)$ ***A1*** ***N3***

20. For multiplying $(\boldsymbol{I} - \boldsymbol{X})(\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2)$ ***M1***

$= \boldsymbol{I}^2 + \boldsymbol{IX} + \boldsymbol{IX}^2 - \boldsymbol{XI} - \boldsymbol{X}^2 - \boldsymbol{X}^3 = \boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2 - \boldsymbol{X} - \boldsymbol{X}^2 - \boldsymbol{X}^3$ ***(A1)(A1)***

$= \boldsymbol{I} - \boldsymbol{X}^3$ ***A1***

$= \boldsymbol{I}$ ***A1***

$\boldsymbol{AB} = \boldsymbol{I} \Rightarrow \boldsymbol{A}^{-1} = \boldsymbol{B}$ ***(R1)***

$(\boldsymbol{I} - \boldsymbol{X})(\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2) = \boldsymbol{I} \Rightarrow (\boldsymbol{I} - \boldsymbol{X})^{-1} = \boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$ ***AG*** ***N0***

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PROGRAMA DEL DIPLOMA DEL BI

SPEC/5/MATHL/HP2/ENG/TZ0/XX

MATHEMATICS
HIGHER LEVEL
PAPER 2

SPECIMEN

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. All students should therefore be advised to show their working.

1. *[Maximum mark: 21]*

The function f is defined on the domain $x \geq 1$ by $f(x) = \frac{\ln x}{x}$.

(a) (i) Show, by considering the first and second derivatives of f, that there is one maximum point on the graph of f.

(ii) State the **exact** coordinates of this point. *[9 marks]*

(iii) The graph of f has a point of inflexion at P. Find the x-coordinate of P. *[3 marks]*

Let R be the region enclosed by the graph of f, the x-axis and the line $x = 5$.

(c) Find the **exact** value of the area of R. *[6 marks]*

(d) The region R is rotated through an angle 2π about the x-axis. Find the volume of the solid of revolution generated. *[3 marks]*

2. *[Maximum mark: 20]*

A farmer owns a triangular field ABC. The side [AC] is 104 m, the side [AB] is 65 m and the angle between these two sides is 60°.

(a) Calculate the length of the third side of the field. *[3 marks]*

(b) Find the area of the field in the form $p\sqrt{3}$, where p is an integer. *[3 marks]*

Let D be a point on [BC] such that [AD] bisects the 60° angle. The farmer divides the field into two parts by constructing a straight fence [AD] of length x metres.

(c) (i) Show that the area of the smaller part is given by $\frac{65x}{4}$ and find an expression for the area of the larger part.

(ii) Hence, find the value of x in the form $q\sqrt{3}$, where q is an integer. *[8 marks]*

(d) Prove that $\frac{\text{BD}}{\text{DC}} = \frac{5}{8}$. *[6 marks]*

3. *[Maximum mark: 29]*

(a) Show that lines $\frac{x-2}{1}=\frac{y-2}{3}=\frac{z-3}{1}$ and $\frac{x-2}{1}=\frac{y-3}{4}=\frac{z-4}{2}$ intersect and find the coordinates of P, the point of intersection. *[8 marks]*

(b) Find the Cartesian equation of the plane Π that contains the two lines. *[6 marks]*

(c) The point Q (3, 4, 3) lies on Π. The line L passes through the midpoint of [PQ]. Point S is on L such that $\left|\overrightarrow{\mathrm{PS}}\right|=\left|\overrightarrow{\mathrm{QS}}\right|=3$, and the triangle PQS is normal to the plane Π. Given that there are two possible positions for S, find their coordinates. *[15 marks]*

4. *[Total maximum mark: 25]*

Part A *[Maximum mark: 13]*

Bag A contains 2 red and 3 green balls.

(a) Two balls are chosen at random from the bag without replacement. Find the probability that 2 red balls are chosen. *[2 marks]*

Bag B contains 4 red and n green balls.

(b) Two balls are chosen without replacement from this bag. If the probability that two red balls are chosen is $\frac{2}{15}$, show that $n=6$. *[4 marks]*

A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.

(c) Calculate the probability that two red balls are chosen. *[4 marks]*

(d) Given that two red balls are chosen, find the probability that a 1 or a 6 was obtained on the die. *[3 marks]*

(This question continues on the next page)

(Question 4 continued)

Part B *[Maximum mark: 12]*

The continuous random variable *X* has probability density function

$$f(x) = \frac{1}{6}x(1+x^2) \quad \text{for } 0 \le x \le 2,$$
$$f(x) = 0 \quad \text{otherwise.}$$

(a) Sketch the graph of f for $0 \le x \le 2$. *[2 marks]*

(b) Write down the mode of *X*. *[1 mark]*

(c) Find the mean of *X*. *[4 marks]*

(d) Find the median of *X*. *[5 marks]*

5. *[Total maximum mark: 25]*

Part A *[Maximum mark: 9]*

Use mathematical induction to prove that $5^n + 9^n + 2$ is divisible by 4, for $n \in \mathbb{Z}^+$. *[9 marks]*

Part B *[Maximum mark: 16]*

Consider the complex geometric series $e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + ...$

(a) Find an expression for *z*, the common ratio of this series. *[2 marks]*

(b) Show that $|z| < 1$. *[2 marks]*

(c) Write down an expression for the sum to infinity of this series. *[2 marks]*

(d) (i) Express your answer to part (c) in terms of $\sin\theta$ and $\cos\theta$.

(ii) Hence show that

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + ... = \frac{4\cos\theta - 2}{5 - 4\cos\theta}$$ *[10 marks]*

MARKSCHEME

SPECIMEN PAPERS

MATHEMATICS

Higher Level

Paper 2

13 pages

Markscheme Instructions

A. Abbreviations

M Marks awarded for attempting to use a correct **Method:** the working must be seen.

(M) Marks awarded for **Method**: this may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**, usually dependent on preceding ***M*** marks: the working must be seen.

(A) Marks awarded for an **Answer** or for **Accuracy**: this may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**

N Marks awarded for **correct** answers, if **no** working (or no relevant working) shown: in general, these will not be all the marks for the question. Examiners should only award these ***N*** marks for correct answers where there is no working (or if there is working which earns no other marks).

B. Using the markscheme

Follow through (ft) marks: Only award **ft** marks when a candidate uses an incorrect answer in a subsequent **part**. Any exceptions to this will be noted on the markscheme. Follow through marks are now the exception rather than the rule within a question or part question. Follow through marks may only be awarded to work that is seen. Do **not** award ***N*** **(ft)** marks. If the question becomes much simpler then use discretion to award fewer marks.
If a candidate mis-reads data from the **question** apply follow-through.

Discretionary (d) marks: There will be rare occasions where the markscheme does not cover the work seen. In such cases, **(d)** should used to indicate where an examiner has used discretion. It must be accompanied by a brief note to explain the decision made.

It is important to understand the difference between “**implied**” marks, as indicated by the brackets, and marks which can only be awarded for work seen - no brackets. The implied marks can only be awarded if **correct** work is seen or implied in subsequent working. Normally this would be in the next line.

Where ***M1 A1*** are awarded on the same line, this usually means ***M1*** for an attempt to use an appropriate formula, ***A1*** for correct substitution.

As ***A*** marks are normally **dependent** on the preceding ***M*** mark being awarded, it is not possible to award ***M0 A1.***

As ***N*** marks are only awarded when there is no working, it is not possible to award a mixture of ***N*** and other marks.

Accept all correct alternative methods, even if not specified in the markscheme Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative (part) solutions, are indicated by **EITHER**….**OR**. Where possible, alignment will also be used to assist examiners to identify where these alternatives start and finish.

Unless the question specifies otherwise, accept **equivalent forms**. On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer The markscheme indicate the required answer, by allocating full marks at that point. Once the correct answer is seen, ignore further working, unless it contradicts the answer.

Brackets will also be used for what could be described as the well-expressed answer, but which candidates may not write in examinations. Examiners need to be aware that the marks for answers should be awarded for the form preceding the brackets eg in differentiating $f(x) = 2\sin(5x-3)$, the markscheme says

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textit{A1}$$

This means that the ***A1*** is awarded for seeing $\left(2\cos(5x-3)\right)5$, although we would normally write the answer as $10\cos(5x-3)$.

As this is an international examination, all **alternative forms of notation** should be accepted.

Where the markscheme specifies ***M2*, *A3***, etc, for an answer do NOT split the marks unless otherwise instructed.

Do **not** award full marks for a correct answer, all working must be checked.

Candidates should be penalized **once IN THE PAPER** for an accuracy error (**AP).** There are two types of accuracy error:

- **Rounding errors:** only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule is *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figure*s.

1. (a) (i) Attempting to use quotient rule $f'(x)=\dfrac{x\frac{1}{x}-\ln x\times 1}{x^2}$ ***(M1)***

$f'(x)=\dfrac{1-\ln x}{x^2}$ ***A1***

$f''(x)=\dfrac{x^2\left(-\frac{1}{x}\right)-(1-\ln x)2x}{x^4}$ ***(M1)***

$f''(x)=\dfrac{2\ln x-3}{x^3}$ ***A1***

Stationary point where $f'(x)=0$, ***M1***

ie $\ln x=1$, (so $x=\mathrm{e}$) ***A1***

$f''(\mathrm{e})<0$ so maximum. ***R1AG*** ***N0***

(ii) Exact coordinates $x=\mathrm{e}, y=\dfrac{1}{\mathrm{e}}$ ***A1A1*** ***N2***

[9 marks]

(b) Solving $f''(0)=0$ ***M1***

$\ln x=\dfrac{3}{2}$ ***(A1)***

$x=\mathrm{e}^{\frac{3}{2}}$ (4.48) ***A1*** ***N2***

[3 marks]

continued ...

Question 1 continued

(c) Area = $\int_1^5 \frac{\ln x}{x} \mathrm{d}x$ ***A1***

EITHER

Finding the integral by substitution/inspection

$u = \ln x, \mathrm{d}u = \frac{1}{x}\mathrm{d}x$ ***(M1)***

$\int u \mathrm{d}u = \frac{u^2}{2}\left(= \frac{(\ln x)^2}{2}\right)$ ***M1A1***

Area = $\left[\frac{(\ln x)^2}{2}\right]_1^5 = \frac{1}{2}\left((\ln 5)^2 - (\ln 1)^2\right)$ ***A1***

Area $= \frac{1}{2}(\ln 5)^2 (= 1.30)$ ***A1*** ***N2***

OR

Finding the integral *I* by parts ***(M1)***

$u = \ln x, \mathrm{d}v = \frac{1}{x} \Rightarrow \mathrm{d}u = \frac{1}{x}, v = \ln x$

$I = uv - \int u \mathrm{d}v = (\ln x)^2 - \int \ln x \frac{1}{x}\mathrm{d}x = (\ln x)^2 - I$ ***M1***

$\Rightarrow 2I = (\ln x)^2 \Rightarrow I = \frac{(\ln x)^2}{2}$ ***A1***

$\Rightarrow$ area $= \left[\frac{(\ln x)^2}{2}\right]_1^5 = \frac{1}{2}\left((\ln 5)^2 - (\ln 1)^2\right)$ ***A1***

Area $= \frac{1}{2}(\ln 5)^2 (= 1.30)$ ***A1*** ***N2***

Note: Award ***N1*** for 1.30 with no working.

[6 marks]

(d) Using $V = \int_a^b \pi y^2 \mathrm{d}x$ ***(M1)***

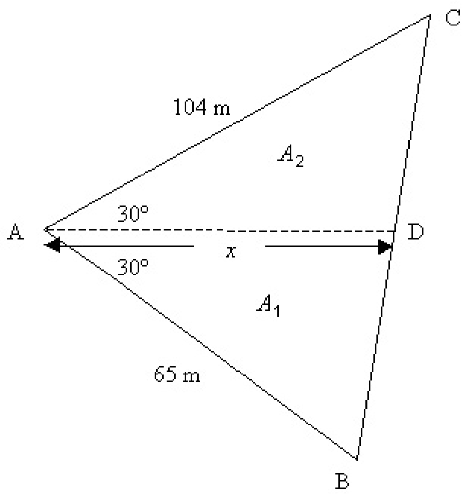
$= \int_1^5 \pi\left(\frac{\ln x}{x}\right)^2 \mathrm{d}x$ ***A1***

$= 1.38$ ***A1*** ***N2***

[3 marks]

Total [21 marks]

2.



(a) Using the cosine rule $\left(a^2 = b^2 + c^2 - 2bc\cos A\right)$ ***(M1)***

Substituting correctly

$\text{BC}^2 = 65^2 + 104^2 - 2(65)(104)\cos 60^\circ$ ***A1***

$= 4225 + 10816 - 6760 = 8281$

$\Rightarrow \text{BC} = 91\text{ m}$ ***A1*** ***N2***

[3 marks]

(b) Finding the area using $= \frac{1}{2}bc\sin A$ ***(M1)***

Substituting correctly, area $= \frac{1}{2}(65)(104)\sin 60^\circ$ ***A1***

$= 1690\sqrt{3}$ (Accept $p = 1690$) ***A1*** ***N2***

[3 marks]

(c) (i) Smaller area $A_1 = \left(\frac{1}{2}\right)(65)(x)\sin 30^\circ$ ***(M1)A1***

$= \frac{65x}{4}$ ***AG*** ***N0***

Larger area $A_2 = \left(\frac{1}{2}\right)(104)(x)\sin 30^\circ$ ***M1***

$= 26x$ ***A1*** ***N1***

(ii) Using $A_1 + A_2 = A$ ***(M1)***

Substituting $\frac{65x}{4} + 26x = 1690\sqrt{3}$ ***A1***

Simplifying $\frac{169x}{4} = 1690\sqrt{3}$ ***A1***

Solving $x = \frac{4 \times 1690\sqrt{3}}{169}$

$\Rightarrow x = 40\sqrt{3}$ (Accept $q = 40$) ***A1*** ***N1***

[8 marks]

continued ...

Question 2 continued

(d) using sin rule in ΔADB and ΔACD ***(M1)***

Substituting correctly $\frac{\text{BD}}{\sin 30^\circ} = \frac{65}{\sin \text{A}\hat{\text{D}}\text{B}} \Rightarrow \frac{\text{BD}}{65} = \frac{\sin 30^\circ}{\sin \text{A}\hat{\text{D}}\text{B}}$ ***A1***

and $\frac{\text{DC}}{\sin 30^\circ} = \frac{104}{\sin \text{A}\hat{\text{D}}\text{C}} \Rightarrow \frac{\text{DC}}{104} = \frac{\sin 30^\circ}{\sin \text{A}\hat{\text{D}}\text{C}}$ ***A1***

Since $\text{A}\hat{\text{D}}\text{B} + \text{A}\hat{\text{D}}\text{C} = 180^\circ$ ***R1***

It follows that $\sin \text{A}\hat{\text{D}}\text{B} = \sin \text{A}\hat{\text{D}}\text{C}$ ***R1***

$\frac{\text{BD}}{65} = \frac{\text{DC}}{104} \Rightarrow \frac{\text{BD}}{\text{DC}} = \frac{65}{104}$ ***A1***

$\Rightarrow \frac{\text{BD}}{\text{DC}} = \frac{5}{8}$ ***AG*** ***N0***

[6 marks]

Total [20 marks]

3. (a) $L_1 : x = 2 + \lambda\, ; y = 2 + 3\lambda\, ; z = 3 + \lambda$ ***(A1)***
$L_2 : x = 2 + \mu\, ; y = 3 + 4\mu\, ; z = 4 + 2\mu$ ***(A1)***
At the point of intersection ***(M1)***
$2 + \lambda = 2 + \mu$ (1)
$2 + 3\lambda = 3 + 4\mu$ (2)
$3 + \lambda = 4 + 2\mu$ (3)
From (1), $\lambda = \mu$. ***A1***
Substituting in (2), $2 + 3\lambda = 3 + 4\lambda$
$\Rightarrow \lambda = \mu = -1$. ***A1***
We need to show that these values satisfy (3). ***(M1)***
They do because LHS = RHS = 2; therefore the lines intersect. ***R1***
So P is $(1, -1, 2)$. ***A1*** ***N3***

[8 marks]

(b) The normal to Π is normal to both lines. It is therefore given by the vector product of the two direction vectors.

Therefore, normal vector is given by $\begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{pmatrix}$ ***M1A1***

$= 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ ***A2***

The Cartesian equation of Π is $2x - y + z = 2 + 1 + 2$ ***(M1)***

i.e. $2x - y + z = 5$ ***A1*** ***N2***

[6 marks]

(c) The midpoint M of [PQ] is (2, 3/2, 5/2). ***M1A1***
The direction of $\overrightarrow{\text{MS}}$ is the same as the normal to Π, ie $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ ***(R1)***

The coordinates of a general point R on $\overrightarrow{\text{MS}}$ are therefore ***(M1)***
$\left(2 + 2\lambda, \frac{3}{2} - \lambda, \frac{5}{2} + \lambda\right)$

It follows that $\overrightarrow{\text{PR}} = (1 + 2\lambda)\boldsymbol{i} + \left(\frac{5}{2} - \lambda\right)\boldsymbol{j} + \left(\frac{1}{2} + \lambda\right)\boldsymbol{k}$ ***A1 A1 A1***

At S, length of $\overrightarrow{\text{PR}}$ is 3, ie ***(M1)***
$(1 + 2\lambda)^2 + (5/2 - \lambda)^2 + (1/2 + \lambda)^2 = 9$ ***A1***
$1 + 4\lambda + 4\lambda^2 + 25/4 - 5\lambda + \lambda^2 + 1/4 + \lambda + \lambda^2 = 9$ ***(A1)***
$6\lambda^2 = \frac{6}{4}$ ***A1***
$\lambda = \pm\frac{1}{2}$ ***A1***
Substituting these values, ***(M1)***
the possible positions of S are $(3, 1, 3)$ and $(1, 2, 2)$ ***A1A1*** ***N2***

[15 marks]

[29 marks]

4. **Part A**

(a) $\mathrm{P}(RR)=\left(\frac{2}{5}\right)\left(\frac{1}{4}\right)$ ***(M1)***

$=\frac{1}{10}$ ***A1*** ***N2***

[2 marks]

(b) $\mathrm{P}(RR)=\frac{4}{4+n}\times\frac{3}{3+n}=\frac{2}{15}$ ***A1***

Forming equation $12\times 5=2(4+n)(3+n)$ ***(M1)***

$12+7n+n^2=90$ ***A1***

$\Rightarrow n^2+7n-78=0$ ***A1***

$n=6$ ***AG*** ***N0***

[4 marks]

(c) **EITHER**

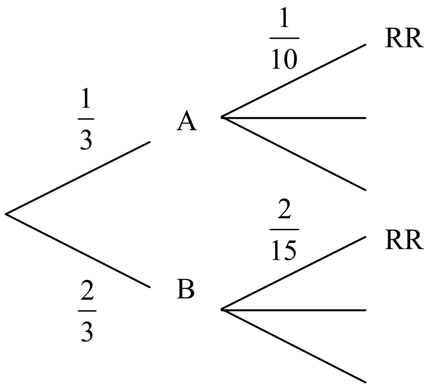
$\mathrm{P}(A)=\frac{1}{3}\qquad \mathrm{P}(B)=\frac{2}{3}$ ***A1***

$\mathrm{P}(\mathrm{RR})=\mathrm{P}(\mathrm{A}\cap\mathrm{RR})+\mathrm{P}(\mathrm{B}\cap\mathrm{RR})$ ***(M1)***

$=\left(\frac{1}{3}\right)\left(\frac{1}{10}\right)+\left(\frac{2}{3}\right)\left(\frac{2}{15}\right)$

$=\frac{11}{90}$ ***A1*** ***N2***

OR



A1

$\mathrm{P}(RR)=\frac{1}{3}\times\frac{1}{10}+\frac{2}{3}\times\frac{2}{15}$ ***M1***

$=\frac{11}{90}$ ***A1*** ***N2***

[3 marks]

continued ...

Question 4 Part A continued

(d) $\mathrm{P}(1 \text{ or } 6) = \mathrm{P}(A)$ ***M1***

$$\mathrm{P}\left(A \middle| RR\right) = \frac{\mathrm{P}(A \cap RR)}{\mathrm{P}(RR)}$$ ***(M1)***

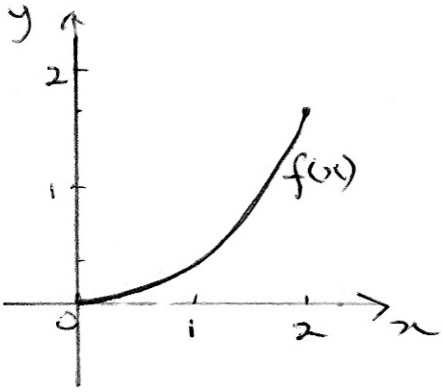
$$= \frac{\left[\left(\frac{1}{3}\right)\left(\frac{1}{10}\right)\right]}{\frac{11}{90}}$$ ***M1***

$$= \frac{3}{11}$$ ***A1*** ***N2***

[4 marks]

Part B

(a)



A2

[2 marks]

(b) Mode = 2 ***A1***

[1 mark]

(c) Using $\mathrm{E}(X) = \int_a^b x\, f(x) \mathrm{d}x$ ***(M1)***

$$\text{Mean} = \frac{1}{6}\int_0^2 (x^2 + x^4) \mathrm{d}x$$ ***A1***

$$= \frac{1}{6}\left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^2$$ ***(A1)***

$$= \frac{68}{45} \ (1.51)$$ ***A1*** ***N2***

[4 marks]

continued ...

Question 4 Part B continued

(d) The median m satisfies $\frac{1}{6}\int_0^m (x + x^3)\mathrm{d}x = \frac{1}{2}$ ***M1*** ***A1***

$\frac{m^2}{2} + \frac{m^4}{4} = 3$ ***(A1)***

$\Rightarrow m^4 + 2m^2 - 12 = 0$

$m^2 = \frac{-2 \pm \sqrt{4+48}}{2} = 2.60555...$ ***(A1)***

$m = 1.61$ ***A1*** ***N3***

[5 marks]

Total [25 marks]

5. **Part A**

Let $f(n) = 5^n + 9^n + 2$ and let P_n be the proposition that $f(n)$ is divisible by 4.

Then $f(1) = 16$ ***A1***

So P_1 is true ***A1***

Let P_n be true for $n = k$ ie $f(k)$ is divisible by 4 ***M1***

Consider $f\left(k+1\right) = 5^{k+1} + 9^{k+1} + 2$ ***M1***

$= 5^k\left(4+1\right) + 9^k\left(8+1\right) + 2$ ***A1***

$= f\left(k\right) + 4\left(5^k + 2 \times 9^k\right)$ ***A1***

Both terms are divisible by 4 so $f\left(k+1\right)$ is divisible by 4. ***R1***

P_k true $\Rightarrow P_{k+1}$ true ***R1***

Since P_1 is true, P_n is proved true by mathematical induction for $n \in \mathbb{Z}^+$. ***R1*** ***N0***

[9 marks]

Part B

(a) $z = z = \frac{1}{2}\mathrm{e}^{2\mathrm{i}\theta} \div \mathrm{e}^{\mathrm{i}\theta}$ ***(M1)***

$z = \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}$ ***A1*** ***N2***

[2 marks]

(b) $\left|z\right| = \frac{1}{2}$ ***A2***

$\left|z\right| < 1$ ***AG***

[2 marks]

(c) Using $S_\infty = \frac{a}{1-r}$ ***(M1)***

$S_\infty = \frac{\mathrm{e}^{\mathrm{i}\theta}}{1 - \frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}}$ ***A1*** ***N2***

[2 marks]

continued ...

Question 5 Part B continued

(d) (i) $S_\infty = \dfrac{e^{i\theta}}{1-\frac{1}{2}e^{i\theta}} = \dfrac{\text{cis}\,\theta}{1-\frac{1}{2}\text{cis}\,\theta}$ ***(M1)***

$\dfrac{\cos\theta + i\sin\theta}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}$ ***(A1)***

Also $S_\infty = e^{i\theta} + \frac{1}{2}e^{2i\theta} + \frac{1}{4}e^{3i\theta} + \ldots\ldots..$

$= \text{cis}\,\theta + \frac{1}{2}\text{cis}\,2\theta + \frac{1}{4}\text{cis}\,3\theta + \ldots\ldots\ldots.$ ***(M1)***

$$S_\infty = \left(\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots\ldots.\right) + i\left(\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + \ldots\right)$$ ***A1***

(ii) Taking real parts,

$$\cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + \ldots = \text{Re}\left(\frac{\cos\theta + i\sin\theta}{1-\frac{1}{2}(\cos\theta + i\sin\theta)}\right)$$ ***A1***

$$= \text{Re}\left(\frac{(\cos\theta + i\sin\theta)}{\left(1-\frac{1}{2}\cos\theta - \frac{1}{2}i\sin\theta\right)} \times \frac{1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta}{\left(1-\frac{1}{2}\cos\theta + \frac{1}{2}i\sin\theta\right)}\right)$$ ***M1***

$$= \frac{\cos\theta - \frac{1}{2}\cos^2\theta - \frac{1}{2}\sin^2\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2 + \frac{1}{4}\sin^2\theta}$$ ***A1***

$$= \frac{\left(\cos\theta - \frac{1}{2}\right)}{1-\cos\theta + \frac{1}{4}\left(\sin^2\theta + \cos^2\theta\right)}$$ ***A1***

$$= \frac{(2\cos\theta - 1)\div 2}{(4-4\cos\theta + 1)\div 4} = \frac{4(2\cos\theta - 1)}{2(5-4\cos\theta)}$$ ***A1***

$$= \frac{4\cos\theta - 2}{5-4\cos\theta}$$ ***A1AG*** ***N0***

[10 marks]

Total [25 marks]

MATHEMATICS
HIGHER LEVEL
PAPER 3

SPECIMEN

1 hour

INSTRUCTIONS TO CANDIDATES

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SECTION A

Statistics and probability

1. *[Maximum mark: 12]*

When a fair die is thrown, the probability of obtaining a '6' is $\frac{1}{6}$.

Charles throws such a die repeatedly.

(a) Calculate the probability that

(i) he throws at least two '6's in his first ten throws;

(ii) he throws his first '6' on his fifth throw;

(iii) he throws his third '6' on his twelfth throw. *[10 marks]*

(b) On which throw is he most likely to throw his first '6'? *[2 marks]*

2. *[Maximum mark: 11]*

In an opinion poll, 540 out of 1200 people interviewed stated that they support government policy on taxation.

(a) (i) Calculate an unbiased estimate of the proportion, p, of the whole population supporting this policy.

(ii) Calculate the standard error of your estimate.

(iii) Calculate a 95 % confidence interval for p. *[9 marks]*

(b) State an assumption required to find this interval. *[2 marks]*

3. *[Maximum mark: 13]*

The 10 children in a class are given two jigsaw puzzles to complete. The time taken by each child to solve the puzzles was recorded as follows.

Child	A	B	C	D	E	F	G	H	I	J
Time to solve Puzzle 1 (mins)	10.2	12.3	9.6	13.8	14.3	11.6	10.5	8.3	9.3	9.9
Time to solve Puzzle 2 (mins)	11.7	12.9	9.9	13.6	16.3	12.2	12.0	8.4	9.8	9.5

(a) For each child, calculate the time taken to solve Puzzle 2 minus the time taken to solve Puzzle 1. *[2 marks]*

(b) The teacher believes that Puzzle 2 takes longer, on average, to solve than Puzzle 1.

(i) State hypotheses to test this belief.

(ii) Carry out an appropriate *t*-test at the 1 % significance level and state your conclusion in the context of the problem. *[11 marks]*

4. *[Maximum mark: 24]*

Let $X_1, X_2, \ldots, X_{12}$ be a random sample from a continuous uniform distribution defined on the interval [0,1]. The random variable *Z* is given by

$$Z = \sum_{n=1}^{12} X_n - 6 .$$

(a) Show that E(*Z*) = 0 and Var(*Z*) = 1. *[6 marks]*

(b) Jim states that *Z* is approximately N(0,1) distributed. Justify this statement. *[2 marks]*

(c) Jim writes a computer program to generate 500 values of *Z*. He obtains the following table from his results.

Range of values of *Z*	Frequency
(−∞,−2)	16
[−2,−1)	66
[−1,0)	180
[0,1)	155
[1,2)	65
[2,∞)	18

(i) Use a chi-squared goodness of fit test to investigate whether or not, at the 5 % level of significance, the N(0,1) distribution can be used to model these results.

(ii) In this situation, state briefly what is meant by

(a) a Type I error;

(b) a Type II error. *[16 marks]*

SECTION B

Sets, relations and groups

1. *[Maximum mark: 6]*

Using deMorgan's laws, prove that $A \Delta B = \left(A \cup B\right) \cap \left(A \cap B\right)'$. *[6 marks]*

2. *[Maximum mark: 11]*

The binary operation $a * b$ is defined by $a * b = \dfrac{ab}{a+b}$, where $a, b \in \mathbb{Z}^{+}$

(a) Prove that $*$ is associative. *[7 marks]*

(b) Show that this binary operation does not have an identity element. *[4 marks]*

3. *[Maximum mark: 16]*

(a) Consider the functions *f* and *g*, defined by

$f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = 5n + 4$,
$g : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ where $g(x, y) = (x + 2y, 3x - 5y)$

(i) Explain whether the function *f* is

(a) injective;

(b) surjective.

(ii) Explain whether the function *g* is

(a) injective;

(b) surjective.

(iii) Find the inverse of *g*. *[13 marks]*

(b) Consider any functions $f : A \to B$ and $g : B \to C$. Given that $g \circ f : A \to C$ is surjective, show that *g* is surjective. *[3 marks]*

4. *[Maximum mark: 11]*

Let the matrix $\boldsymbol{T}$ be defined by $\begin{pmatrix} x & x+2 \\ x-5 & -x \end{pmatrix}$ such that $\det \boldsymbol{T} = 1$.

(a) (i) Show that the equation for x is $2x^2 - 3x - 9 = 0$.

(ii) The solutions of this equation are a and b, where $a > b$. Find a and b. *[5 marks]*

(b) Let $\boldsymbol{A}$ be the matrix where $x = 3$

(i) Find $\boldsymbol{A}^2$.

(ii) Assuming that matrix multiplication is associative, find the smallest group of 2×2 matrices which contains $\boldsymbol{A}$**,** showing clearly that this is a group. *[6 marks]*

5. *[Maximum mark: 16]*

The group $(G, \times)$ has a subgroup $(H, \times)$. The relation R is defined on G $(x\,R\,y) \Leftrightarrow (x^{-1}y \in H)$, for $x,\ y \in G$.

(a) Show that R is an equivalence relation. *[8 marks]*

(b) Given that $G = \{e, p, p^2, q, pq, p^2q\}$, where e is the identity element, $p^3 = q^2 = e$, and $qp = p^2q$, prove that $qp^2 = pq$. *[3 marks]*

(c) Given also that $H = \{e, p^2q\}$, find the equivalence class with respect to R which contains pq. *[5 marks]*

SECTION C

Series and differential equations

1. *[Maximum mark: 6]*

Calculate $\lim_{x \to 0}\left(\frac{1}{x} - \frac{1}{\sin x}\right)$. *[6 marks]*

2. *[Maximum mark: 6]*

Use the integral test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent for $p > 1$. *[6 marks]*

3. *[Maximum mark: 24]*

(a) (i) Find the first four derivatives with respect to x of $y = \ln(1 + \sin x)$

(ii) Hence, show that the Maclaurin series, up to the term in x^4, for y is

$$y = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$$ *[10 marks]*

(b) Deduce the Maclaurin series, up to and including the term in x^4, for

(i) $y = \ln(1 - \sin x)$;

(ii) $y = \ln \cos x$;

(iii) $y = \tan x$. *[10 marks]*

(c) Hence calculate $\lim_{x \to 0}\left(\frac{\tan(x^2)}{\ln \cos x}\right)$. *[4 marks]*

4. *[Maximum mark: 24]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{xy}{4-x^2}=1$, where $|x|<2$ and $y=1$ when $x=0$.

(a) Use Euler's method with $h = 0.25$, to find an approximate value of y when $x=1$, giving your answer to two decimal places. *[10 marks]*

(b) (i) By first finding an integrating factor, solve this differential equation. Give your answer in the form $y = f(x)$.

(ii) Calculate, correct to two decimal places, the value of y when $x = 1$. *[10 marks]*

(c) Sketch the graph of $y = f(x)$ for $0 \le x \le 1$. Use your sketch to explain why your approximate value of y is greater than the true value of y. *[4 marks]*

SECTION D

Discrete mathematics

1. *[Maximum mark: 11]*

(a) Write the number $10\,201$ in base 8. *[4 marks]*

(b) Prove that if a number is divisible by 7 that the sum of its base 8 digits is also divisible by 7. *[5 marks]*

(c) Using the result of part (b), show that the number $10\,201$ is not divisible by 7. *[2 marks]*

2. *[Maximum mark: 13]*

Let a and b be two positive integers.

(a) Show that $\gcd(a,b)\times\operatorname{lcm}(a,b)=ab$ *[6 marks]*

(b) Show that $\gcd(a,a+b)=\gcd(a,b)$ *[7 marks]*

3. *[Maximum mark: 6]*

Find the remainder when 67^{101} is divided by 65. *[6 marks]*

4. *[Maximum mark: 6]*

Solve the system of linear congruences

$$x\equiv 1\left(\bmod 3\right);\ x\equiv 2\left(\bmod 5\right);\ x\equiv 3\left(\bmod 7\right).$$ *[6 marks]*

5. *[Maximum mark: 10]*

The matrix below is the adjacency matrix of a graph H with 6 vertices A, B, C, D, E, F.

$$\begin{array}{c|cccccc} & \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{F} \\ \text{A} & 0 & 1 & 0 & 1 & 1 & 0 \\ \text{B} & 1 & 0 & 1 & 0 & 0 & 1 \\ \text{C} & 0 & 1 & 0 & 1 & 1 & 0 \\ \text{D} & 1 & 0 & 1 & 0 & 0 & 1 \\ \text{E} & 1 & 0 & 1 & 0 & 0 & 1 \\ \text{F} & 0 & 1 & 0 & 1 & 1 & 0 \end{array}$$

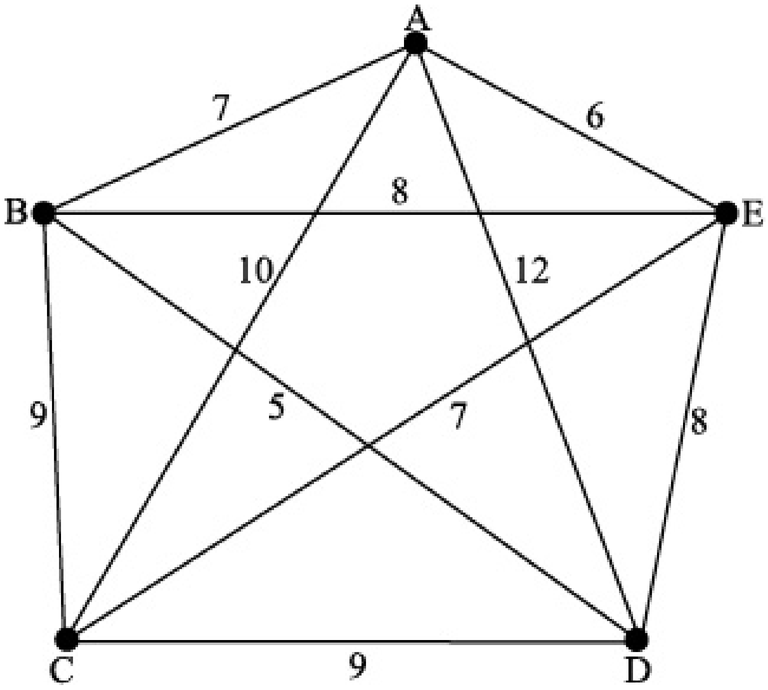
(a) Show that H is not planar. *[3 marks]*

(b) Find a planar subgraph of H by deleting one edge from it. *[3 marks]*

(c) Show that any subgraph of H (excluding H itself) is planar. *[4 marks]*

6. *[Maximum mark: 14]*

Let G be the graph below.



(a) Find the total number of Hamiltonian cycles in G, starting at vertex A. Explain your answer. *[3 marks]*

(b) (i) Find a minimum spanning tree for the subgraph obtained by deleting A from G. *[3 marks]*

(ii) Hence, find a lower bound for the travelling salesman problem for G. *[3 marks]*

(c) Give an upper bound for the travelling salesman problem for the graph above. *[2 marks]*

(d) Show that the lower bound you have obtained is not the best possible for the solution to the travelling salesman problem for G. *[3 marks]*

MARKSCHEME

SPECIMEN PAPERS

MATHEMATICS

Higher Level

Paper 3

18 pages

Markscheme Instructions

A. Abbreviations

M Marks awarded for attempting to use a correct **Method:** the working must be seen.

(M) Marks awarded for **Method**: this may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**, usually dependent on preceding ***M*** marks: the working must be seen.

(A) Marks awarded for an **Answer** or for **Accuracy**: this may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**

N Marks awarded for **correct** answers, if **no** working (or no relevant working) shown: in general, these will not be all the marks for the question. Examiners should only award these ***N*** marks for correct answers where there is no working (or if there is working which earns no other marks).

B. Using the markscheme

Follow through (ft) marks: Only award **ft** marks when a candidate uses an incorrect answer in a subsequent **part**. Any exceptions to this will be noted on the markscheme. Follow through marks are now the exception rather than the rule within a question or part question. Follow through marks may only be awarded to work that is seen. Do **not** award ***N*** **(ft)** marks. If the question becomes much simpler then use discretion to award fewer marks.
If a candidate mis-reads data from the **question** apply follow-through.

Discretionary (d) marks: There will be rare occasions where the markscheme does not cover the work seen. In such cases, **(d)** should used to indicate where an examiner has used discretion. It must be accompanied by a brief note to explain the decision made.

It is important to understand the difference between "**implied**" marks, as indicated by the brackets, and marks which can only be awarded for work seen - no brackets. The implied marks can only be awarded if **correct** work is seen or implied in subsequent working. Normally this would be in the next line.

Where ***M1 A1*** are awarded on the same line, this usually means ***M1*** for an attempt to use an appropriate formula, ***A1*** for correct substitution.

As ***A*** marks are normally **dependent** on the preceding ***M*** mark being awarded, it is not possible to award ***M0 A1.***

As ***N*** marks are only awarded when there is no working, it is not possible to award a mixture of ***N*** and other marks.

Accept all correct alternative methods, even if not specified in the markscheme Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc.* Other alternative (part) solutions, are indicated by **EITHER**….**OR**. Where possible, alignment will also be used to assist examiners to identify where these alternatives start and finish.

Unless the question specifies otherwise, accept **equivalent forms**. On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer The markscheme indicate the required answer, by allocating full marks at that point. Once the correct answer is seen, ignore further working, unless it contradicts the answer.

Brackets will also be used for what could be described as the well-expressed answer, but which candidates may not write in examinations. Examiners need to be aware that the marks for answers should be awarded for the form preceding the brackets eg in differentiating $f(x) = 2\sin(5x-3)$, the markscheme says

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right) \qquad \textit{A1}$$

This means that the ***A1*** is awarded for seeing $\left(2\cos(5x-3)\right)5$, although we would normally write the answer as $10\cos(5x-3)$.

As this is an international examination, all **alternative forms of notation** should be accepted.

Where the markscheme specifies ***M2*, *A3***, etc, for an answer do NOT split the marks unless otherwise instructed.

Do **not** award full marks for a correct answer, all working must be checked.

Candidates should be penalized **once IN THE PAPER** for an accuracy error (**AP).** There are two types of accuracy error:

- **Rounding errors:** only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule is *unless otherwise stated in the question all numerical answers must be given exactly or to three significant figure*s.

SECTION A

Statistics and probability

Note: Values obtained from a GDC may differ slightly from those obtained from tables.

1. (a) (i) Number of 6s obtained is $\text{B}\left(10, \frac{1}{6}\right)$. ***(M1)***

Prob (at least 2) $= 1 - \left(\frac{5}{6}\right)^{10} - 10\left(\frac{5}{6}\right)^{9}\left(\frac{1}{6}\right)$ ***(A1)***

$= 0.515$ ***A1*** ***N3***

(ii) We require the first 4 throws not to be 6s followed by a 6 on the 5th throw. ***(M1)***

$\text{Prob} = \left(\frac{5}{6}\right)^{4} \times \left(\frac{1}{6}\right)$ ***(A1)***

$= 0.0804$ ***A1*** ***N3***

(iii) If he throws his third 6 on the X^{th} throw, X has a negative binomial distribution. ***(R1)***

$\text{P}(X = 12) = \binom{11}{2} \times \left(\frac{5}{6}\right)^{9} \times \left(\frac{1}{6}\right)^{3}$ ***(M1)(A1)***

$= 0.0493$ ***A1*** ***N4***

[10 marks]

(b) Probability of 1st six on n^{th} throw $= \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6}$ ***M1***

This is a decreasing function so most likely throw is the first. ***A1*** ***N1***

[2 marks]

Total [12 marks]

2. (a) (i) Estimated proportion $= \frac{540}{1200} (= 0.45)$ ***(M1)A1*** ***N2***

(ii) Estimated standard error $= \sqrt{\frac{540 \times 660}{1200^3}}$ ***M1A1***

$= 0.0144$ ***A1*** ***N1***

(iii) 95 % confidence limits are $\frac{540}{1200} \pm 1.96\sqrt{\frac{540 \times 660}{1200^3}}$ ***(M1)(A1)***

$= 0.45 \pm 1.96 \times 0.0144$ ***(A1)***

The 95 % confidence interval is $[0.422, 0.478]$ ***A1*** ***N4***

[9 marks]

continued ...

Question 2 continued

(b) **EITHER**
The sample needs to be random. ***R2***

OR
We can approximate a binomial distribution by a normal distribution. ***R2***

[2 marks]

Total [11 marks]

3. (a) The values are

Child	A	B	C	D	E	F	G	H	I	J
difference	1.5	0.6	0.3	-0.2	2.0	0.6	1.5	0.1	0.5	-0.4

A2

[2 marks]

(b) (i) $\mathrm{H}_0 : \mu_1 = \mu_2$: $\mathrm{H}_1 : \mu_1 < \mu_2$ ***A1A1***

(ii) **EITHER**

$\sum d = 6.5$: $\sum d^2 = 9.77$

$\hat{\sigma}^2 = \frac{9.77}{9} - \frac{6.5^2}{90}$ ***(M1)(A1)***

$= 0.6161111$ ***A1***

$t = \frac{\frac{6.5}{10}}{\sqrt{\frac{0.6161111}{10}}}$ ***(M1)***

$= 2.62$ ***A1***

Degrees of freedom = 9 ***A1***

Critical value = 2.82 ***A1***

OR

$p = 0.0139$ ***A7***

THEN

Insufficient evidence to support the teacher's belief ***R1***
that puzzle 2 takes longer than puzzle 1. ***R1***

[11 marks]

Total [13 marks]

4. (a) $\mathrm{E}(X)=\frac{1}{2}$, $\mathrm{Var}(X)=\frac{1}{12}$ ***A1A1***

$\mathrm{E}(Z)=\sum_{i=1}^{12}\mathrm{E}(X_n)-6$ ***(M1)***

$=12\times\frac{1}{2}-6$ ***A1***

$=0$ ***AG*** ***N0***

$\mathrm{Var}(Z)=\sum_{i=1}^{12}\mathrm{Var}(X_n)$ ***(M1)***

$=12\times\frac{1}{12}$ ***A1***

$=1$ ***AG*** ***N0***

[6 marks]

(b) Since n is reasonably large, ***R1***
the central limit theorem ensures that Z is approximately normal. ***R1***

[2 marks]

(c) (i)

Range of values of z	Observed frequency	Expected frequency	
$(-\infty,-2)$	16	**11.35**	***(A1)***
$[-2,-1)$	66	**68.00**	***(A1)***
$[-1,0)$	180	**170.65**	***(A1)***
$[0,1)$	155	**170.65**	***(A1)***
$[1,2)$	65	**68.00**	***(A1)***
$[2,\infty)$	18	**11.35**	***(A1)***

$\chi^2=\frac{(16-11.35)^2}{11.35}+\ldots$ ***(M1)***

$=7.94$ ***A1***

Degrees of freedom = 5 ***A1***

Critical value $=11.07$ ***A1***

We conclude that the data fit the $\mathrm{N}(0,1)$ distribution. ***R1***

at the 5% level of significance ***A1***

(ii) (a) Type I error concluding that the data do not fit $\mathrm{N}(0,1)$ when in fact they do. ***R2***

(b) Type II error concluding that data fit $\mathrm{N}(0,1)$ when in fact they do not. ***R2***

[16 marks]

Total [24 marks]

SECTION B

Sets, relations and groups

1. $A \Delta B = (A \setminus B) \cup (B \setminus A)$

$= (A \cap B') \cup (B \cap A')$

$= ((A \cap B') \cup B) \cap ((A \cap B') \cup A')$ ***M1A1***

$= ((A \cup B) \cap (B' \cup B)) \cap ((A \cup A') \cap (B' \cup A'))$ ***M1A1***

$= ((A \cup B) \cap U) \cap (U \cap (B' \cup A'))$ ***A1***

$= (A \cup B) \cap (A' \cup B')$

$= (A \cup B) \cap (A \cap B)'$ ***A1***

Note: Illustration using a Venn diagram is not a proof.

[6 marks]

2. (a) $(a*b)*c = \left(\frac{ab}{a+b}\right)*c$ ***(M1)***

$$= \frac{\frac{abc}{a+b}}{\frac{ab}{a+b}+c}$$ ***A1***

$$= \frac{abc}{ab+ac+bc}$$ ***A1***

$a*(b*c) = a*\left(\frac{bc}{b+c}\right)$ ***(M1)***

$$= \frac{\frac{abc}{a+b}}{a+\frac{bc}{b+c}}$$ ***A1***

$$= \frac{abc}{ab+ac+bc}$$ ***A1***

$\therefore (a*b)*c = a*(b*c)$ ***R1***

so $*$ is associative. ***AG***

[7 marks]

(b) Suppose e is an identity element, then $e*a = a*e = a$ ***(M1)***

$\frac{ea}{e+a} = a$ ***A1***

$ea = ea + a$ ***M1***

ea cancels on both sides so there is no solution for *e*. ***R1***

i.e. no identity element ***AG***

[4 marks]

Total [11 marks]

3. (a) (i) (a) f is an increasing function **_R1_**

so it is injective. **_A1_**

(b) Let $f(n)=1$ (or any other appropriate value) **_M1_**

Then $5n+4=1$, $n=-\frac{3}{5}$ which is not in the domain

$\therefore$ f is not surjective. **_A1_**

(ii) $g(x,y)=(x+2y,\, 3x-5y)$ so $\begin{pmatrix}1 & 2\\ 3 & -5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x+2y\\ 3-5y\end{pmatrix}$

METHOD 1

(a) Let $g(x,y)=g(s,t)$ so $(x+2y,\, 3x-5y)=(s+2t,\, 3s-5t)$ **_M1_**

$x+2y=s+2t,\ 3x-5y=3s-5t$ **_M1_**

$y=t$ and $x=s$ $\Rightarrow (x,y)=(s,t)$

g is injective. **_A1_**

(b) Let (u,v) be an element of the codomain.

$x+2y=u,\ 3x-5y=v$ **_M1_**

Then $-11y=-3u+v$ so $y=\frac{3u-v}{11}$ **_A1_**

and $11x=5u+2v$ so $x=\frac{5u+2v}{11}$ **_A1_**

Since $\left(\frac{5u+2v}{11}, \frac{3u-v}{11}\right)$ is in the domain then g is surjective. **_R1_**

continued ...

Question 3 (a) (ii) continued

METHOD 2

(a) $\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix}$ ***M1***

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ t \end{pmatrix}$ since $\det\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix} \neq 0$, ***A1***

$(x, y) = (s, t)$

g is injective. ***A1***

(b) Let (u, v) be an element of the codomain.

$\begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$ ***M1***

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -5 \end{pmatrix}^{-1}\begin{pmatrix} u \\ v \end{pmatrix}$ ***A1***

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 5 & 2 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ ***A1***

Since $\left(\frac{5u+2v}{11}, \frac{3u-v}{11}\right)$ is in the domain then g is surjective. ***R1***

(iii) $g^{-1}(x, y) = \left(\frac{5x+2y}{11}, \frac{3x-y}{11}\right)$ ***A2***

[13 marks]

(b) $g \circ f$ is surjective, so for every $z \in \mathbb{C}$ there exists $x \in A$ such that

$(g \circ f)(x) = z$ (ie $g(f(x)) = z$) ***R1***

Let $y = f(x) \in B$. ***R1***

For every $z \in \mathbb{C}$ there exists $y \in B$ such that $g(y) = z$. ***R1***

$\therefore$ g is surjective. ***AG***

[3 marks]

Total [16 marks]

4. (a) (i) $\det \boldsymbol{T} = x(-x) - (x+2)(x-5)$ ***M1***

$= -x^2 - x^2 + 3x + 10$ ***A1***

$1 = -2x^2 + 3x + 10$ ***(A1)***

$0 = 2x^2 - 3x - 9$ ***AG*** ***N0***

(ii) $0 = (2x+3)(x-3)$

$x = -\frac{3}{2}$ or $x = 3$

$a = 3,\ b = -\frac{3}{2}$ ***A1 A1*** ***N2***

[5 marks]

(b) (i) $\boldsymbol{A} = \begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix} \Rightarrow \boldsymbol{A}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ ***A1***

(ii) $\boldsymbol{A}^3 = \begin{pmatrix} -3 & -5 \\ 2 & 3 \end{pmatrix}$ ***A1***

$\boldsymbol{A}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (= \boldsymbol{I})$ ***A1***

$\boldsymbol{A}^2$ is a self-inverse ***A1***

$\boldsymbol{A}^3 = \boldsymbol{A}^{-1}$. ***A1***

$\therefore$ the set $\{\boldsymbol{A}, \boldsymbol{A}^2, \boldsymbol{A}^3, \boldsymbol{I}\}$ is closed under matrix multiplication;

has an identity $\boldsymbol{I}$; is associative and each element has an inverse.
Therefore it is a group. ***R1AG*** ***N0***

[6 marks]

Total [11 marks]

5. (a) $x^{-1}x = e \in H$. ***M1***

$\Rightarrow x R x \Rightarrow R$ is reflexive ***R1***

$x R y \Rightarrow x^{-1}y \in H$

$\Rightarrow \left(x^{-1}y\right)^{-1} \in H$ ***A1***

$x^{-1}y\left(x^{-1}y\right)^{-1} = e$

so $\left(x^{-1}y\right)^{-1} = y^{-1}x$ ***A1***

$\Rightarrow y^{-1}x \in H \Rightarrow y R x \Rightarrow R$ is symmetric ***R1***

$x R y$ and $y R z \Rightarrow x^{-1}y \in H$ and $y^{-1}z \in H$

$\therefore$ $\left(x^{-1}y\right)\left(y^{-1}z\right) \in H$ since H is closed. ***A1***

$x^{-1}\left(y y^{-1}\right)z \in H$

$x^{-1}z \in H$ ***A1***

$\Rightarrow x R z \Rightarrow R$ is transitive. ***R1***

$\therefore$ R is an equivalence relation. ***AG***

[8 marks]

(b) $p^3 = q^2 = e \qquad qp = p^2q$

$qp^2 = (qp)p$

$= \left(p^2q\right)p$ ***A1***

$= p^2(qp)$

$= p^2\left(p^2q\right)$ ***A1***

$= p^3(pq)$ ***A1***

$= pq$ ***AG***

[3 marks]

(c) $H = \left\{e, p^2q\right\}$

$y R pq \Rightarrow y^{-1}pq = e \Rightarrow pq = y$ ***A1***

or $y^{-1}pq = p^2q \Rightarrow pq = yp^2q$

$pq^2 = yp^2q^2$ ***A1***

$p = yp^2$

$p^2 = yp^3$ ***A1***

$p^2 = y$ ***A1***

$\therefore$ The equivalence class is $\left\{p^2,\ pq\right\}$ ***A1***

[5 marks]

Total [16 marks]

SECTION C

Series and differential equations

1. Let $f(x)=\dfrac{\sin x-x}{x\sin x}$ ***(M1)***

$\lim\limits_{x\to 0} f(x)=\lim\limits_{x\to 0}\left(\dfrac{\cos x-1}{\sin x+x\cos x}\right)$ ***A1A1***

$=\lim\limits_{x\to 0}\left(\dfrac{-\sin x}{2\cos x-x\sin x}\right)$ ***A1A1***

$=0$ ***A1*** ***N2***

[6 marks]

2. For $p>1$, $\dfrac{1}{x^p}$ is

positive for $x\geq 1$, and decreasing for $x\geq 1$. ***A1A1***

$\lim\limits_{L\to\infty}\int_1^L \dfrac{1}{x^p}\mathrm{d}x=\lim\limits_{L\to\infty}\left[\dfrac{1}{(1-p)x^{p-1}}\right]_1^L$ ***(M1)***

$=\lim\limits_{L\to\infty}\dfrac{1}{(1-p)L^{p-1}}-\dfrac{1}{1-p}$ ***A1***

$=\dfrac{1}{p-1}$ ***A1***

The convergence of this integral ensures the convergence of the series using the integral test. ***R1AG*** ***N0***

[6 marks]

3. (a) (i) $y=\ln(1+\sin x)$

$y'=\dfrac{\cos x}{1+\sin x}$ ***A1***

$y''=-\dfrac{1}{1+\sin x}$ ***A1***

$y^{(3)}=\dfrac{\cos x}{(1+\sin x)^2}$ ***A1***

$y^{(4)}=\dfrac{-\sin x(1+\sin x)^2-2(1+\sin x)\cos^2 x}{(1+\sin x)^4}$ ***(M1)A1***

(ii) $y(0)=0\,;y'(0)=1$ ***A1A1***

$y''(0)=-1\,;y^{(3)}(0)=1\,;y^{(4)}(0)=-2$ ***A1A1***

A1

$\ln(1+\sin x)=x-\dfrac{1}{2}x^2+\dfrac{1}{6}x^3-\dfrac{1}{12}x^4+\ldots$ ***AG*** ***N0***

[10 marks]

(b) (i) $\ln(1-\sin x) = \ln\left(1+\sin\left(-x\right)\right)$ ***(M1)***

$= -x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$ ***A1*** ***N2***

(ii) $\ln(1+\sin x) + \ln(1-\sin x) = \ln(1-\sin^2 x)$ ***(M1)***

$= \ln\cos^2 x$ ***A1***

So $\ln\cos^2 x = -x^2 - \frac{1}{6}x^4 + \ldots$ ***A1***

$\ln\cos x = -\frac{1}{2}x^2 - \frac{1}{12}x^4 + \ldots$ ***A1*** ***N2***

(iii) Differentiating, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\cos x\right) = \frac{1}{\cos x} \times \left(-\sin x\right)$ ***(M1)***

$= -\tan x$ ***A1***

$\tan x = x + \frac{1}{3}x^3 + \ldots$ ***A2*** ***N3***

Note: No term in x^4 since $\tan(-x) = -\tan x$

[10 marks]

(c) $\frac{\tan\left(x^2\right)}{\ln\cos x} = \frac{x^2 + \frac{x^4}{3} + \ldots}{-\frac{x^2}{2} - \frac{x^4}{12} + \ldots}$ ***(M1)***

$= \frac{1 + \frac{x^4}{3} + \ldots}{-\frac{1}{2} - \frac{x^2}{12} + \ldots}$ ***A1***

$\to -2$ as $x \to 0$ ***A1***

so $\lim_{x\to 0}\left(\frac{\tan(x^2)}{\ln\cos x}\right) = -2$ ***A1*** ***N3***

[4 marks]

Total [24 marks]

4. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 - \frac{xy}{4 - x^2}$

x	y	dy/dx	$h \times$ dy/dx	
0	1	1	0.25	***A2***
0.25	1.25	0.9206349206	0.2301587302	***A2***
0.5	1.48015873	0.8026455027	0.2006613757	***A2***
0.75	1.680820106	0.6332756132	0.1583189033	***A2***
1	1.839139009			***A1***

To two decimal places, when $x = 1, y = 1.84$. ***A1*** ***N0***

[10 marks]

(b) (i) Integrating factor $= \mathrm{e}^{\int\left(\frac{x}{4-x^2}\right)\mathrm{d}x}$ ***(M1)***

$= \mathrm{e}^{\left(-\frac{1}{2}\ln(4-x^2)\right)}$ ***A1***

$= \frac{1}{\sqrt{4-x^2}}$ ***A1***

It follows that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{\sqrt{4-x^2}}\right) = \frac{1}{\sqrt{4-x^2}}$ ***(M1)***

$\frac{y}{\sqrt{4-x^2}} = \arcsin\left(\frac{x}{2}\right) + C$ ***A1A1***

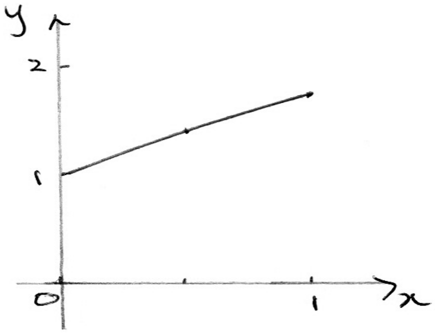
Putting $x = 0, y = 1, \Rightarrow \frac{1}{2} = C$ ***A1***

Therefore, $y = \sqrt{4-x^2}\left(\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}\right)$ ***A2*** ***N0***

(ii) When $x = 1, y = 1.77$. ***A1*** ***N1***

[10 marks]

(c)



A2

Since $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing the value of y is over-estimated at each step. ***R1A1***

[4 marks]

Total [24 marks]

SECTION D

Discrete mathematics

1. (a) $10201 = a\times 8^4 + b\times 8^3 + c\times 8^2 + d\times 8 + e$ ***M1***

$= 4096a + 512b + 64c + 8d + e \Rightarrow a = 2$ ***A1***

$10201 - 2\times 4096 = 2009 = 512b + 64c + 8d + e \Rightarrow b = 3$

$2009 - 3\times 512 = 473 = 64c + 8d + e \Rightarrow c = 7$

$473 - 7\times 64 = 25 = 8d + e \qquad \Rightarrow d = 3$ and $e = 1$

$10201 = 23731$ (base 8) ***A2*** ***N2***

[4 marks]

(b) $8^n \equiv 1 \pmod 7$ for positive integer n ***A1***

Consider the octal number

$u_n u_{n-1}..u_1 u_o = u_n + u_{n-1} + u_1 + u_0 \pmod 7$ ***(M1)***

from which it follows that an octal number is divisible by 7 if and only if ***A1***
the sum of the digits is divisible by 7. ***R1***

Hence $10201 \equiv a + b + c + d + e \pmod 7$ ***A1***

[5 marks]

(c) $10201 \equiv 2 + 3 + 7 + 3 + 1 \equiv 2 \pmod 7$ ***A2***

[2 marks]

Total [11 marks]

2. (a) Let $p_1,...,p_n$ be the set of primes that divide either a or b ***M1***

Then $a = p_1^{\alpha_1} p_2^{\alpha_2}...p_n^{\alpha_n}$ and $b = p_1^{\beta_1} p_2^{\beta_2}...p_n^{\beta_n}$ ***A1A1***

Hence $ab = p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2}...p_n^{\alpha_n+\beta_n}$ ***A1***

Furthermore $\min\left\{\alpha_j,\beta_j\right\} + \max\left\{\alpha_j,\beta_j\right\} = \alpha_j + \beta_j$ for $j = 1,2,...,n$ ***A1***

Hence $ab = p_1^{\min\{\alpha_1,\beta_1\}+\max\{\alpha_1,\beta_1\}}...p_n^{\min\{\alpha_n,\beta_n\}+\max\{\alpha_n,\beta_n\}}$ ***A1***

$ab = \gcd(a,b)\times \text{lcm}(a,b)$ ***AG***

[6 marks]

(b) $\gcd(a,b)\,|\,a$ and $\gcd(a,b)\,|\,b$ ***A1***

Hence $\gcd(a,b)\,|\,a+b$ ***A1***

so that $\gcd(a,b)\,|\gcd(a,a+b)$ * ***A1***

Also $\gcd(a,a+b)\,|\,a$ and $\gcd(a,b)\,|\,a+b$ ***A1***

Hence $\gcd(a,a+b)\,|\,b$ ***A1***

so that $\gcd(a,a+b)\,|\gcd(a,b)$ ** ***A1***

From * and ** : $\gcd(a,b) = \gcd(a,a+b)$ ***A1 AG***

[7 marks]

Total [13 marks]

3. $67^{101} \equiv 2^{101} \pmod{65}$ ***A1***

$2^6 \equiv -1 \pmod{65}$ ***(M1)***

$2^{101} \equiv \left(2^6\right)^{16} \times 2^5$ ***A1***

$\equiv \left(-1\right)^{16} \times 32 \pmod{65}$ ***A1***

$\equiv 32 \pmod{65}$ ***A1***

$\therefore$ remainder is 32 ***A1*** ***N2***

[6 marks]

4. $x \equiv 1 \pmod{3} \Rightarrow x = 3k + 1$ ***A1***

Choose k such that $3k + 1 \equiv 2 \pmod{5}$ ***M1***

With Euclid's algorithm or otherwise we find

$k \equiv 7 + 5h$ ***A1***

Choose h such that $22 + 15k \equiv 3 \pmod{7}$ ***M1***

With Euclid's algorithm or otherwise

$k \equiv 2 + 7j$ ***A1***

Hence $x = 22 + 15(2 + 7j) = 52 + 105j$ ***A1*** ***N3***

[6 marks]

5. (a) H is not planar because if it were then $e \le 2v - 4$ ***M1***

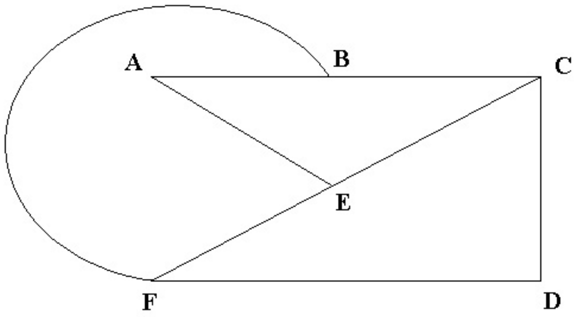
But here $e = 9$ and $v = 6$ ***A1***

And hence the inequality is not satisfied ***A1***

So H is not planar ***AG*** ***N0***

[3 marks]

(b) Deleting the edge connecting A with D we can draw the graph as below



M1A1

which shows that it is planar. ***A1***

[3 marks]

continued ...

Question 5 continued

(c) The adjacency matrix can also be written as:

$$\begin{array}{c|cccccc} & A & C & F & B & D & E \\ \hline A & 0 & 0 & 0 & 1 & 1 & 1 \\ C & 0 & 0 & 0 & 1 & 1 & 1 \\ F & 0 & 0 & 0 & 1 & 1 & 1 \\ B & 1 & 1 & 1 & 0 & 0 & 0 \\ D & 1 & 1 & 1 & 0 & 0 & 0 \\ E & 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

M1A1

Hence with a suitable permutation of the last three rows and of the last three columns the general case can be reduced to part (b). ***R1A1***
Any subgraph of H (excluding H itself) is planar ***AG***

[4 marks]

Total [10 marks]

6. (a) Starting from vertex A there are 4 choices. From the next vertex there are three choices, etc… ***M1R1***
So the number of Hamiltonian cycles is $4! = 24$. ***A1*** ***N1***

[3 marks]

(b) (i) Start (for instance) at B, using Prim's algorithm
Then D is the nearest vertex ***M1***
Next E is the nearest vertex ***A1***
Finally C is the nearest vertex
So a minimum spanning tree is $B \to D \to E \to C$ ***A1*** ***N1***

[3 marks]

(ii) A lower bound for the travelling salesman problem is then obtained by adding the weights of AB and AE to the weight of the minimum ***M1***
spanning tree (ie 20) ***A1***
A lower bound is then $20 + 7 + 6 = 33$ ***A1*** ***N1***

[3 marks]

continued ...

Question 6 continued

(c) A minimum spanning tree for *G* would be

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & A \arrow[r] & E \arrow[r] & C \\ & D \end{tikzcd}$$

of weight 26 ***A1***

Thus an upper bound is given by $26 \times 2 = 52$ ***A1***

[2 marks]

(d) Eliminating C from *G* a minimum spanning tree is $E \to A \to B \to D$ ***M1***
of weight 18 ***A1***
Adding BC to CE(18+9+7) gives a lower bound of $34 > 33$ ***A1***
So 33 not the best lower bound ***AG*** ***N0***

[3 marks]

Total [14 marks]